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IMPROVED THIRD ORDER RUNGE-KUTTA METHODS BASED ON THE CONVEX COMBINATION OF VARIETY OF MEAN

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ABSTRACT

This paper presents the construction of a 3-stage explicit Runge Kutta method of order three based on the convex combination of Arithmetic mean, Harmonic mean and Heronian mean which was recently constructed base on the linear combination of Arithmetic mean, Geometric mean and Harmonic mean as against the constructions based on Arithmetic mean, Heronian mean, Contra-Harmonic mean, Harmonic mean and Geometric mean. The region of absolute stability of the new method was established. Numerical experiments were performed on some initial value problems and the results showed that the new method compete well enough when compared with some existing 3-stage explicit Runge Kutta methods.

Keywords: Explicit Runge-Kutta, Arithmetic Mean, Harmonic Mean, Heronian Mean, Absolute Stability.

INTRODUCTION

There are many mathematical models in science and engineering that take the form of ordinary differential equations subject to an initial condition. Some of these differential equations don't have analytical solutions, for this reason approximate solutions through numerical methods are important. Among the existing numerical methods, Runge Kutta method is one of the best commonly used numerical method for solving an initial value problem (IVP) hence the growing interest in its exploration in the recent years. In the development of methods for solving ordinary differential equations it is not clear that the arithmetic mean is always the best choice (Sanugi B, 1986). Naturally the arithmetic mean formulae are the most convenient to use but are not necessarily the most accurate formulae to use for all types of problems (Sanugi B,1986). This has been shown in recent years by researchers. There are, however, several other types of 'mean' which have also produced consistent approximations. See ([1], [2], [4], [5], [6], [9], [10], [12], [13], [14] and [15]). Evans [6] developed a new 4th order Runge-Kutta method based on the contra-harmonic mean. The developed scheme was used to solve some problems and they asserted that it is accurate when compared to its equivalents. Also in [Evans and Yaacob, 1995], they constructed a new 4th Order Runge-Kutta method based on the Heronian Formula in which they used Heronian mean in the derivation. They went further to compare the scheme with several Runge-Kutta methods of 4th order based on variety of mean. A new 4th order embedded method based on the harmonic mean was constructed by Yaacob N. [Yaacob and Sanugi B., 1995] where harmonic mean was embedded in the arithmetic mean viewed Runge-kutta methods. The proposed method was found accurate and cost effective compared to the classical methods. Ponalagusamy R. [Ponalagusamy and Chandra, 2011] developed a new 5th order method based on Heronian mean to compute numerical solution of IVP in ODE. They took modest effort to examine the suitability, adaptability and accuracy of the method. Based on their numerical results, they observed that the new scheme is superior compared to existing methods including the 5th order Runge-Kutta methods based on arithmetic mean, geometric mean, harmonic mean and Contra-Harmonic mean. A 3-stage geometric explicit Runge-Kutta methods for singular autonomous initial value problems in ordinary differential equations was developed by Akanbi M.A. [Akanbi, 2011] in which geometric mean was incorporated in the classical 3-stage Runge-Kutta methods. The scheme showed that it is stable, efficient and accurate when compared with some other conventional methods. Third order harmonic mean for autonomous initial value problem was constructed by Wusu A.S. [Wusu *et al.,* 2012]. The method was derived based on harmonic mean and was confirmed to be better than any third order of any form of explicit Runge-Kutta methods. Wusu A.S. in [Wusu *et al.,* 2012] took a step forward to develop a 4-Stage Harmonic Explicit Runge-Kutta methods. The proposed scheme was compared with some existing schemes. Olaniyan A.S. [Olaniyan *et al.,* 2020] constructed a 2-Stage heronian Implicit Runge-Kutta methods. The paper was found to perform better than the classical 2-Stage Implicit Runge-Kutta methods. Recently, Rini [Rini Yanti *et al.,* 2014] considered a third order Runge-Kutta method based on a linear combination of arithmetic mean, harmonic mean and geometric mean. They confirmed that their method is suitable for studying first order Initial Value Problems when compared with other existing third other Runge Kutta methods. This idea was extended to fourth order by Bazuaye F. E. [Bazuaye, 2019] where a new fourth order hybrid Runge-Kutta method based on linear combination of arithmetic mean, geometric mean and harmonic mean was constructed. However, our aim in this paper is to construct a third order Runge-Kutta methods based on a convex combination in [Khattri, 2012]. We will also establish the stability analysis and computational experiment by comparing our method with some existing methods in the literature.

MATERIALS AND METHODS

Consider an initial value problem (IVP), which can be written in the form:

$$y' = f(x, y(x)), \qquad y(x_0) = y_0 \tag{1}$$

As stated by Lambert [Lambert J. D. (1991)], the general R-Stage Runge Kutta method to solve (1) is given as follows:

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$$y_{n+1} - y_n = h\varphi(x_n, y_n, h)$$

$$\varphi(x_n, y_n, h) = \sum_{r=1}^{R} c_r k_r$$

$$k_1 = f(x, y)$$

$$k_r = f\left(x + ha_r, y + h\sum_{s=1}^{r-1} b_{rs}k_s\right)$$

A classical third order Runge Kutta method constructed from the above is given by:

$$y_{n+1} - y_n = \frac{h}{4}(k_1 + 2k_2 + k_3)$$

where

$$k_1 = f(x_n, y_n) \quad (2)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \quad (3)$$

$$k_3 = (x_n + h, y_n - hk_1 + hk_2) \quad (4)$$

Which was expressed in an arithmetic mean based formula as:

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2}\right) \quad (5)$$

Researcher modified (5) by replacing it with different types of "mean" such as geometric mean (GM):

$$y_{n+1} = y_n + \frac{h}{2}\left(\sqrt{k_1k_2} + \sqrt{k_2k_3}\right), \quad (6)$$

Harmonic mean (HM):

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3}\right), \quad (7)$$

Heronian mean:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + k_3 + \sqrt{k_1k_2} + \sqrt{k_2k_3}\right). \quad (8)$$

However, Rini et al [11] replaced (5) with a linear combination of arithmetic mean, harmonic mean and geometric mean given as:

$$R_m(k_1, k_2) = \frac{14AM(k_1k_2) - HM(k_1k_2) + 32GM(k_1k_2)}{45} \quad (9)$$

to produce

$$y_{n+1} = y_n + \frac{h}{90}\left(7(k_1 + 2k_2 + k_3) - \left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right) + 32\left(\sqrt{k_1k_2} + \sqrt{k_2k_3}\right)\right)$$

with

$$k_1 = f(x_n, y_n) \quad (10)$$

$$k_2 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1\right) \quad (11)$$

$$k_3 = \left(x_n + \frac{2}{3}h, y_n - \frac{4}{9}hk_1 + \frac{10}{9}hk_2\right) \quad (12)$$

Construction of the Proposed Scheme

We intend to derive our method based on the convex combination in [7] given as:

$$He(x, y) = (1-p)G(x, y) + pA(x, y), \ \ 0 \le p \le 1 \quad (13)$$

which is used to produce

$$R_m(k_1, k_2) = \frac{48HE(k_1k_2) - HM(k_1k_2) - 2AM(k_1k_2)}{45} \quad (14)$$

Replacing (5) with (14), we have:

$$y_{n+1} = y_n + \frac{h}{90}\left(16\left(k_1 + 2k_2 + k_3 + \sqrt{k_1k_2} + \sqrt{k_2k_3}\right) - \left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right) - (k_1 + 2k_2 + k_3)\right) \quad (15)$$

with

$$k_1 = f(y_n) \quad (16)$$

$$k_2 = f(\mathrm{y_n} + \mathrm{hb_1k_1}) \quad (17)$$

$$k_3 = (y_n + hb_2k_1 + hb_3k_2) \quad (18)$$

Expanding k_2 and k_3, we have

$$k_2 = f + hb_2ff_y + \frac{h^2}{2}b_1^2f^2f_{yy} + 0(h)^3 \quad (19)$$

and

$$k_3 = f + h(b_1 + b_2)ff_y + \frac{h^2}{2}[(b_2 + b_3)^2 + 2b_1b_3]f^2f_{yy} + 0(h)^3 \quad (20)$$

Evaluating the terms in (15) such that

$$\sqrt{k_1k_2} = f + \frac{h}{2}b_1ff_y + \frac{h^2}{4}\left(b_1^2f^2f_{yy} - \frac{h}{2}b_1^2ff_y^2\right), \quad (21)$$

$$\sqrt{k_2k_3} = f + \frac{h}{2}(b_1 + b_2 + b_3)ff_y - \frac{h^2}{8}((b_1^2 + (b_2 + b_3)^2 - 2b_1(b_2 + b_3))ff_y^2 + \frac{1}{4}(b_1^2 + (b_2 + b_3)^2)f^2f_{yy}, \quad (22)$$

$$\frac{k_1k_2}{k_1+k_2} = f + \frac{h}{2}b_1ff_y + \frac{h^2}{4}\left(b_1^2f^2f_{yy} - b_1^2ff_y^2\right), \quad (23)$$

$$\frac{k_2k_3}{k_2+k_3} = f + \frac{h}{2}(b_1 + b_2 + b_3)ff_y + h^2((b_1b_3 - \frac{1}{2}b_2b_3 - \frac{1}{4}b_1^2 + \frac{1}{4}b_2^2 - \frac{1}{4}b_3^2 + \frac{1}{4}b_1b_2)ff_y^2 + (\frac{1}{4}b_1^2 + \frac{1}{4}b_3^2 + \frac{1}{4}b_2^2 + \frac{1}{2}b_2b_3)f^2f_{yy})$$

Substituting (16), (19) - (24) in (15); simplifying and collecting like terms, (15) then becomes

$$y_{n+1} = y_n + hf + h^2\left(\frac{1}{2}b_1 + \frac{1}{4}b_2 + \frac{1}{4}b_3\right)ff_y + h^2\left(\frac{1}{6}b_2b_3 + \frac{1}{3}b_1^2 + \frac{1}{12}b_2^2 + \frac{1}{12}b_3^2\right)f^2f_{yy} + h^3\left(\frac{3}{8}b_1b_2 + \frac{15}{8}b_1b_3 + \frac{9}{4}b_2b_3 - \frac{3}{4}b_1^2 - \frac{3}{8}b_2^2 - \frac{3}{8}b_3^2\right)ff_y^2 \quad (25)$$

Comparing the coefficients of h, h^2 and h^3 in (25) with the Taylor's series expansion which is of the form:

$$y_{n+1} = y_n + hf + \frac{h^2}{2!}ff_y + \frac{h^3}{3!}\left(ff_y^2 + f^2f_{yy}\right) \quad (26)$$

Then we have the following set of nonlinear system of equation:

$$ff_y : \frac{1}{2}b_1 + \frac{1}{4}b_2 + \frac{1}{4}b_3 = \frac{1}{2}$$

$$f^2f_{yy} : \frac{1}{6}b_2b_3 + \frac{1}{3}b_1^2 + \frac{1}{12}b_2^2 + \frac{1}{12}b_3^2 = \frac{1}{6} \quad (27)$$

$$ff_y^2 : \frac{3}{8}b_1b_2 + \frac{15}{8}b_1b_3 + \frac{9}{4}b_2b_3 - \frac{3}{4}b_1^2 - \frac{3}{8}b_2^2 - \frac{3}{8}b_3^2 = \frac{1}{6}$$

Upon solving (27), we obtain $b_1 = \frac{1}{2}$, $b_2 = -\frac{1}{12}$ and $b_3 = \frac{13}{12}$. Substituting these values i.e. b_1, b_2 and b_3 in (17) and (18) then the proposed scheme becomes

$$y_{n+1} = y_n + \frac{h}{90}\left(16\left(k_1 + 2k_2 + k_3 + \sqrt{k_1k_2} + \sqrt{k_2k_3}\right) - \left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right) - (k_1 + 2k_2 + k_3)\right) \tag{28}$$

with

$$k_1 = f(y_n) \tag{29}$$

$$k_2 = f\left(y_n + \frac{1}{2}hk_1\right) \tag{30}$$

$$k_3 = \left(y_n - \frac{1}{12}hk_1 + \frac{13}{12}hk_2\right) \tag{31}$$

Stability Analysis of the Proposed Scheme

To obtain the stability region of our method, we will make use of the Dalquist test problem $y' = \lambda y$ on (16) - (18) and subsequently expanded to get:

$$k_1 = \lambda y \tag{32}$$
$$k_2 = \lambda y + \frac{1}{2}h\lambda^2 y \tag{33}$$
$$k_3 = \lambda y + \frac{1}{2}h\lambda^2 y + \frac{13}{24}h^2\lambda^3 y \tag{34}$$

Substituting (32) - (34) in (15), simplifying the resulting expression, letting $z = h\lambda$ and ignoring the terms for which z^t with $t > 3$, we have the stability polynomial of our scheme as:

$$\frac{y_{n+1}}{y_n} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \tag{35}$$

We obtain the region of absolute stability of the proposed method using Matlab package as follows:

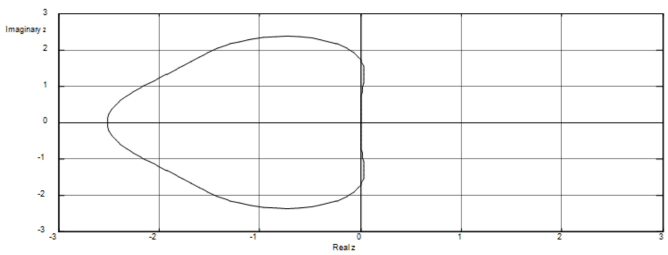

Figure 1: Region of Absolute Stability of the Proposed Method

Numerical Experiment

In this section, we will evaluate some problems which were considered in [11] using different step sizes to show the accuracy and suitability of the proposed scheme. The approximate solutions of the proposed method (RKCC) is compared with the approximate solutions of Runge Kutta method based on arithmetic mean (RKAM), Runge Kutta method based on geometric mean (RKGM), Runge Kutta method based on harmonic mean (RKHM) and Runge Kutta method based on the linear combination of arithmetic mean, geometric mean and harmonic mean (RKMC).

Consider the following IVPs:

1. $y'(x) = \frac{1}{y}$, $y(0) = 1$ with the exact solution $y(x) = \sqrt{2x+1}$ on [0, 1].

Solving with different step sizes: $h = 0.2$ and $h = 0.1$, the results are expressed in the figures below

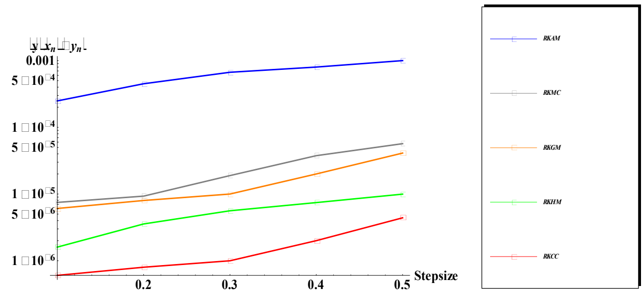

Figure 2 : Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.

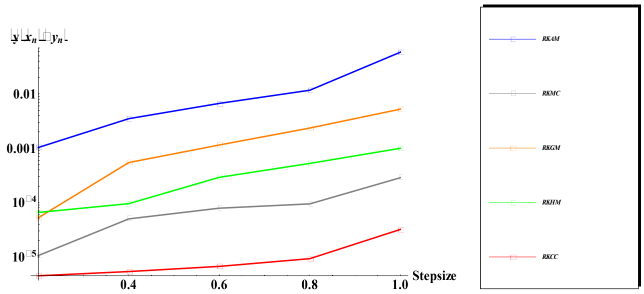

Figure 3 : Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.2

2. $y'(x) = \frac{1}{1+x^2} - 2y^2$, $y(0) = 0$ with the exact solution $y(x) = \frac{1}{1+x^2}$ on [0, 1].

Solving with different step sizes: $h = 0.1$ and $h = 0.01$, the results are expressed in the figures below

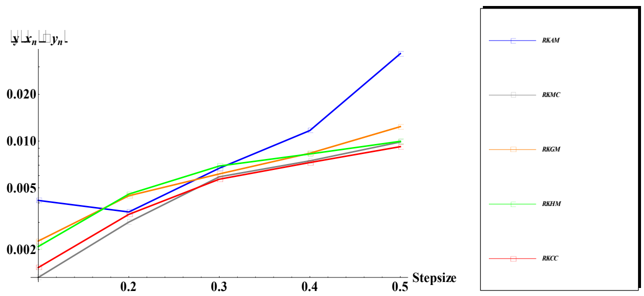

Figure 4: Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.1

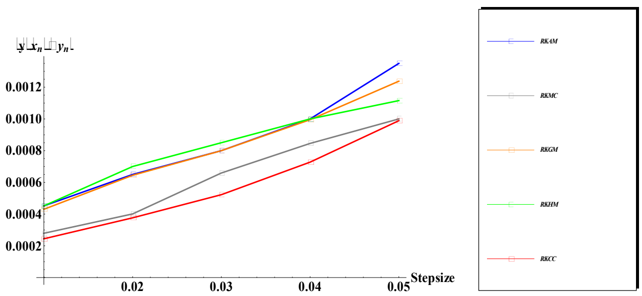

Figure 5: Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.01

3. $y'(x) = y^2(\ln x)^3 - 2xy(\ln x)^4 + 2\ln x + 2$, $y(1) = 0$ with the exact solution $y(x) = 2x\ln x$ on [1, 2].

Solving with different step sizes: $h = 0.1$ and $h = 0.01$, the results are expressed in the figures below

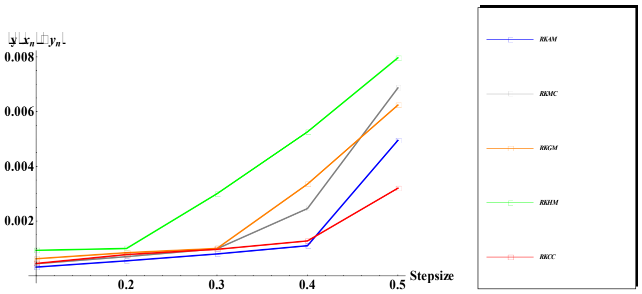

Figure 6: Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.1

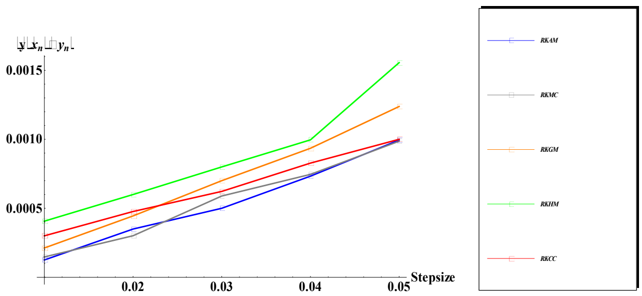

Figure 7: Graph of RKCC, RKMC, RKGM, RKHM and RKAM with h = 0.01

Remark

From the Tables above, it is observed that the proposed RKCC method compete well enough when compared with Runge Kutta method based on arithmetic mean (RKAM), Runge Kutta method based on geometric mean (RKGM), Runge Kutta method based on harmonic mean (RKHM) and Runge Kutta method based on linear combination of arithmetic mean, geometric mean and harmonic mean (RKMC).

CONCLUSION

This paper presents the construction of 3-stage explicit Runge kutta method of order three based on the convex combination of arithmetic mean, harmonic mean and heronian mean as against the constructions based on arithmetic mean, geometric mean, harmonic mean, heronian mean, contra-harmonic mean and recently based on the linear combination of arithmetic mean, harmonic mean and geometric mean. We established the constructed method's region of absolute stability which is the same with that of the classical 3-stage explicit Runge kutta methods. We went further to test the accuracy and suitability of our method by using it to solve some initial value problems. From the results obtained via the numerical experiments, it is evident that our method can be used as an alternative to some existing methods mentioned in the literature.

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