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COMPARISON BETWEEN CURVATURE AND NORMAL CURVATURE ON SMOOTH LOGICALLY CARTESIAN SURFACE MESHES USING MATLAB

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ABSTRACT

In mathematics, the differential geometry of surfaces deals with the differential geometry of smooth surfaces with various additional structures, most often, a Riemannian metric. Surfaces have been extensively studied from various perspectives: extrinsically, relating to their embedding in Euclidean space and intrinsically, reflecting their properties determined solely by the distance within the surface as measured along curves on the surface. One of the fundamental concepts investigated is the Gaussian curvature, Surfaces naturally arise as graphs of functions of a pair of variables, and sometimes appear in parametric form or as loci associated to space curves. The aim of this paper is to compare between Curvature and Normal Curvature on Smooth Logically Cartesian Surface Meshes using Matlab And we found In case 1 , case 2 and case 3 figures show the main results between two sheets curvature and one sheet curvature when we adjust the parameter in 3 Axis vectors as follows in case 1 shown that the parameters as two faces curvature , In case 2 shown that the parameter of case 2 as a positive face of curvature (upper curvature) or one sheet curvature and case 3 processed the parameters in negative way of curvature (lower curvature) or one sheet curvature . As for the normal curve shown the sample data in positive direction of plan that means is approximately near to one sheet positive direction.

Keywords: Comparison ,Curvature , Normal Curvature , Smooth Logically Cartesian Surface Meshes , Matlab.

INTRODUCTION

Differential geometry is a discipline of mathematics that uses the techniques of calculus and linear algebra to study problems in geometry. The theory of plane, curves and surfaces in the three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th and the 19th century. Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds, [2, 4, 6] Differential geometry is closely related to differential topology and the geometric aspects of the theory of differential equations. Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces[10]. Curvature formulas for parametrically defined curves and surfaces are well-known both in the classical literature on Differential Geometry and in the contemporary literature on Geometric Modeling . Curvature formulas for implicitly defined curves and surfaces are more scattered and harder to locate [15]. The curvatures of a smooth surface are local measures of its shape. Here we consider analogous quantities for discrete surfaces, meaning triangulated polyhedral surfaces. Often the most useful analogs are those which preserve integral relations for curvature, like the Gauss/Bonnet theorem or the force balance equation for mean curvature. For simplicity, we usually restrict our attention to surfaces in Euclidean three-spaces IR^3 [9].

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PLANE CURVES

Now let's consider in particular plane curves $(n = 2)$. We equip R^2 with the standard orientation and let J denote the counterclockwise rotation by 90° so that $J(e_1) = e_2$and for any vector $v, J(v)$ is the perpendicular vector of equal length such that $\{v, Jv\}$ is an oriented basis. Given a (regular smooth) plane curve α, its (unit) normal vector [DE: Normal eneinhe its vector] N is defined as $N(s) = J(T(s))$. Since $\vec{k} = T'$ is perpendicular to , it is a scalar multiple of N. Thus we can define the (signed) curvature $k_g\ of\ \ \alpha\ k_g N = \vec{k}\ \ (so\ that\ k_g = \pm|\vec{k}| = \pm k$. For an arbitrary regular parameterization of α, we find

$$k_g = \frac{\det(\dot{\alpha}, \ddot{\alpha})}{|\dot{\alpha}|^2}$$

From $N \perp T\ and\ T' = k_g N$ we see immediately that $N' = -k_g T$We can combine these equations as

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k_g \\ -k_g & 0 \end{pmatrix}\begin{pmatrix} T \\ N \end{pmatrix}$$

Rotating ortho normal frame, infinitesimal rotation (speed k_g given by skew-symmetric matrix. The curvature tells us how fast the tangent vector T turns as we move along the curve at unit speed. Since $T(s)$ is a unit vector in the plane, it can be expressed as $(\cos\theta, \sin\theta)$ for some $\theta = \theta(s)$. Although θ is not uniquely determined (but only up to a multiple of 2π) we claim that we can make a smooth choice of θ along the whole curve. Indeed, if there is such a θ, its derivative is $\theta' = k_g$ Picking any θ_0 such that $T(0) = (\cos\theta_0, \sin\theta_0)$ define $\theta(s) \coloneqq \theta_0 + \int_0^s k_g(d)ds$ This lets us prove what is often called the fundamental theorem of plane curves [DE: Hauptsatz der lokalen Kurventheorie] (although it really doesn't seem quite that important): Given a smooth function $k_g = I \to IR$ there exists a smooth unit-speed curve $\alpha : I \to IR$ with signed curvature k_g this curve is unique up to rigid motion. First note that integrating k_g gives the angle function $\theta : I \to IR$ (uniquely up to a constant of integration), or equivalently gives the tangent vector $T = (cos\,\theta, sin\,\theta)$ (uniquely up to a rotation). Integrating T then gives α (uniquely up to a vector constant of integration, that is, up to a translation). [7]

CHARACTERIZING PLANE CURVES

A plane curve is a curve which is contained in a two-dimensional plane. This section will look to describe a plane curve as a function of how much the curve is bending at points along the curve. For a plane curve the amount of bending experienced at each point is a scalar value called curvature. A plane curve can be determined up to rigid transformations by its curvature. Much of the current work on shape analysis uses a limited number of landmarks to describe the shape. Using only a limited number of landmarks may well result in a large amount of useful information being lost. Furthermore, if the shapes lie in different areas of space, Procrustes methods using landmarks are required to align the shapes. The technique of using curvature to analyse shapes offers an alternative to these approaches. Curvature can be calculated over the whole curve which limits the amount of information about the shape which is lost [6].

CURVES

Definition (IV.A):

Many plane curves can be described as the graph of a function $f : [a, b] \to R$. But such a simple curve as a plane circle cannot And for space curves, it is obvious that one has to find other means of description: We let again $IR^2\text{resp.}\ IR^3$denote ordinary $2-, resp. 3-dimensional$ vector spaces, equipped with orthonormal bases $\{i, j\}, resp. \{i, j, k\}$[13]. One of the most important tools used to analyze a curve is the Frenet frame, which is a moving frame that provides a coordinate system at each point of the curve that is "best adapted" to the curve near that point. Different space curves are only distinguished by the way in which they bend and twist and quantitatively measured by the differential geometric invariants called curvature and torsion of the curve [3].

Definition (IV. B)

A curve is called regular if it is never stationary. In other words, the speed is always positive, or the velocity never vanishes [11].

CURVATURE

Definition (V.A):

Well, a line is not curved at all; its curvature has to be zero. A circle with a small radius is more "curved" than a circle with a large radius. Circles and lines have constant curvature. Curves that are not (pieces of) circles or lines will have a curvature varying from point to point [5].

Definition (V.B):

let $\alpha: I \to R^3$ be a curve parametrized by are length $s\epsilon I$ the number $|\alpha''(s)| = k(s)\ is\ called\ curvature$ [14].

SMOOTH CURVES

Our goal is to define a notion of curvature and torsion for discrete curves We will compare our discrete notions to those of the classical (smooth) differential geometry and as such to parametrized curves $s: IR \to IR^3$. We will always assume s to be sufficiently differentiable [4].

TOPOLOGICAL SURFACES

We mostly interested in smooth regular surfaces defined in the following section. However few times we will use the following general definition. A connected subset Σ in the Euclidean space IR^3 is called a topological surface (more precisely an embedded surface without boundary) if any point of $p\epsilon\Sigma$ admits a neighborhood W in Σ that can be parameterized by an open subset in the Euclidean plane; that is, there is an injective continuous map $U \to W$ from an open set $U \subset R^2$ such that its inverse $W \to U$ is also continuous [2].

THE GEOMETRY OF SURFACES

We have an idea of what a surface is, how do we detect its geometry? One of the most important techniques in mathematics and, indeed, all of the natural sciences, is that of linear approximation. By this we mean the following. We recognize that the nonlinear or curved object at hand is too complicated to study directly, so we approximate it by something linear: a line, a plane, a Euclidean space. We then study the linear object and, from it, infer results about the original curved object. [8].

CURVATURE DISPLAYING

Methods of displaying curvature include normal vectors, contour lines and color. Normal vectors can indicate the surface curvature by a length proportional to the radius of curvature. Contour lines include reference plane and a series of equally spaced planes parallel to it. The intersection of these planes with the surface results in planar curves on the surface. These curves can aid in determining the surface features, e.g. saddle points appear as passes and maxima and minima appear as encircled .Displaying curvature variation as color variation is used scale in which the minimum curvature value corresponds to one end of the color spectrum and the maximum curvature value to the other end of the spectrum, with a linear distribution in between. Color change represents a percent change in curvature, i.e. a logarithmic color scale [12].

CURVATURES OF SMOOTH SURFACES

Given a (two-dimensional, oriented) surface M (smoothly immersed) in $\boldsymbol{E^3}$we understand its local shape by looking at the Gauß map $\nu : M \to S^2$ given by the unit normal vector $\nu = \nu_p$ at each point $\in M$. Its derivative at p is a linear map from $T_pM\ \ to\ T_{\nu p}S^2$ But these spaces are naturally identified, being parallel planes in $\boldsymbol{E^3}$so we can view the derivative as an endomorphism $-S_p: T_pM \to T_pM$ The map S_p is called the shape operator (or Weingarten map). The shape operator is the complete second-order invariant (or curvature) which determines the original surface M . Usually, however, it is more convenient not to work with the operator S_p but instead with scalar quantities. Its eigenvalues $k_1\ \ and\ \ k_2$ of S_p are called principal curvatures, and (since they cannot be globally distinguished) it is their symmetric functions which have the most geometric meaning.

We define the Gauss curvature $K = k_1 k_2$ as the determinant of S_p and the mean curvature $H = \frac{k_1 + k_2}{2}$ as its trace. Note that the sign of H depends on the choice of unit normal , and so often it is more natural to work with the vector mean curvature (or mean curvature vector) $H = Hv$ Note furthermore that some authors use the opposite sign on S_p and thus H, and many us $H = \frac{k_1 + k_2}{2}$ justifying the name mean curvature [9].

NORMAL CURVATURE

We now tackle the problem of defining curvature on regular surfaces. We do this by expressing it in terms of the curvatures of regular curves on the surface. We start with the notion of normal curvature, which is defined with respect to a given regular curve on a surface.

Definition (XI.A)

Let C be a regular curve on regular surface C passing through point $p \in S$, k be the curvature of C at p, n be the unit normal vector to C at p, N be the unit normal vector to the surface at p, and $cos\ \theta = < n, N >$, Then the normal curvature of C at point p is defined to be the signed quantity$k_n = k\cos\theta$. At first glance, this definition doesn't seem terribly useful, since we'd like a definition of curvature that only depends on the properties of the surface, independent of the curves we can draw on it. It turns out that the definition of normal curvature is independent of the specific choice of curve C and only depends on the value of its tangent at point p. To do this, we first express the normal curvature in terms of the differential of the Gauss map. $N\ o\ C$denotes the restriction of the Gauss map to the curve C. Since the normal vector $N\ (p)$ is orthogonal to every tangent vector at p, $< N\ (p), C'(0) > = < N\ C(0), C'(0) > = 0$ Differentiating both sides yields $< (N\ oC)'(0), C'(0) > = < N\ (p), C''(0) >$.Thus,

$$\begin{aligned} k_n &= k(0)\cos\theta \\ &= k(0) < n(0), N\left(C(0)\right) > \\ &= < C''(0), N\left(C(0)\right) > \\ &= - < C'(0), (N\ oC)'(0) > \\ &= - < C'(0), dN_p(C'(0)) > \end{aligned}$$

The last line thus shows that the normal curvature only depends on the tangent $C'(0)$ This development is very reassuring, since it gives us a notion of the curvature of a surface in a specific direction in the tangent plane at p, namely $C'(0)$. A natural next step would be to determine the directions of minimum and maximum normal curvature, and if a minimum and maximum even exist. It turns out that they do, and that they are the negative eigenvalues of the differential of the Gauss map at point p [1].

Theorem (XI.B)

Let S be a regular surface and $p \in S$. Let k_1 , k_2 be the minimum and maximum normal curvatures at p and e_i their associated principal directions. Then let v be some unit vector in R^2 . Then for some $\theta \in [0, 2\pi), v = e_1\cos\theta + e_2\sin\theta$ and the normal curvature in the direction v is given by

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta. \quad \textbf{[1]}$$

Example (XI.C)

(Hyperboloid of Two Sheets Curvature).
Let M denote the hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

parameterized by

$$x(u, v) = (a\ \ sinh\ u\ cos\ v, b\ sinh\ u\ sin\ v, e\ cosh\ u).$$

Then

$$\begin{aligned} x_u &= (a\ cosh\ u\ cos\ v, b\ cosh\ u\ sin\ v, e\ sinh\ u) \\ x_v &= (-a\ sinh\ u\ sin\ v, b\ sinh\ u\ cos\ v, 0) \end{aligned}$$

and

$$\begin{aligned} &x_u \times x_v \\ &= (-bc\ \sinh^2 u\cos v\,, -ac\ \sinh^2 \sin v\ \ , ab\sinh u\cosh v) \end{aligned}$$

Dividing by $|x_u \times x_v|$

$$U = \frac{x_u \times x_v}{W}$$

where

$$W = \sqrt{b^2c^2\sinh^4 u\cos^2 v + a^2c^2\sinh^4\sin^2 u\cos^2 v + a^2b^2\sinh^2 u\cosh^2 u}$$

We then have

$$\begin{aligned} E &= a^2\cosh^2 u\cos^2 v + b^2\cosh^2 u\sin^2 v + c^2\sinh^2 u \\ F &= -a^2 sinh\ u\ cosh\ u\ sin\ v\ co\ s\ v \\ &\quad + b^2\ sinh\ u\ cosh\ u\ sin\ v\ cos\ v \\ G &= a^2\sinh^2 u\sin^2 v + b^2\sinh^2 u\cos^2 v \end{aligned}$$

with

$$\begin{aligned} EG - F^2 &= b^2c^2\sinh^4 u\cos^2 v + a^2c^2\sinh^4\sin^2 u\cos^2 v \\ &\quad + a^2b^2\sinh^2 u\cosh^2 u \\ &= W^2 \end{aligned}$$

The following second partials then give $I, m\ and\ n$.

$$\begin{aligned} x_{uu} &= (a\ sinh\ u\ cos\ v, b\ sinh\ u\ sin\ v, c\ cosh\ u) \\ x_{uv} &= (-a\ cosh\ u\ sin\ v, b\ cosh\ u\ cos\ v, 0) \\ x_{uu} &= (-a\ sinh\ u\ cos\ v, -b\ sinh\ u\ sin\ v, 0) \\ I &= x_{uu}.U \end{aligned}$$

$$= \frac{-abc\sinh^3 u\cos^2 v - abc\sinh^3 u\sin^2 v +\ abc\ sinh\ u\ \cosh^2 u}{W}$$

$$= \frac{abc\ sinh\ u}{W}$$

using$1\ + \sinh^2 u = \cosh^2 u$

$$m = x_{uv}.U$$

$$= \frac{abc\sinh^2 u\cosh u\ sin\ v\ cos\ v - abc\sinh^2 u\ \ cosh\ u\ sin\ v\ cos\ v}{W}$$

$$= 0$$

$$n = x_{vv}.U$$

$$= \frac{abc\sinh^3 u\cos^2 v + abc\sinh^3 u\sin^2 v}{W}$$

$$= \frac{abc\sinh^3 u}{W}$$

Hence, we obtain the Gauss curvature

$$K = \frac{In - m^2}{EG - F^2} = \frac{a^2b^2c^2\sinh^4 u}{W^4}$$

which we may write as

$$K = \frac{1}{\left(\frac{W^2}{abc\sinh^2 u}\right)^2}$$

where

$$\frac{W^2}{abc\sinh^2 u} = \frac{bc}{a}\sinh^2 u\cos^2 v + \frac{ac}{b}\sinh^2 u\sin^2 v + \frac{ab}{c}\cosh^2 u$$

Now, the coordinate functions of the parametrization are $x = a\, sinh\, u\, cos\, v$, $y = b\, sinh\, u\, sin\, v$ and $z = c\, cosh\, u$, so the reader can check that the Gauss curvature may be written in terms of $x, y\ and\ z$ as

$$K = \frac{1}{a^2 b^2 c^2 \left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]^2}. \quad [8]$$

MATLAB FOR EXAMPLE

```
clc;
clear all;
[X,Y,Z] = meshgrid (-10:0.5:10,-10:0.5:10,-10:0.5:10);
a=1;
b=1;
c=1;
M = - X.^2/a^2 - Y.^2/b^2 + Z.^2/c^2;
p=patch(isosurface(X,Y,Z,M,1));
set(p,'FaceColor','black','EdgeColor','none');
daspect([1 1 1])
view(3);
camlight
```

Input:

Case 1: meshgrid (-10:0.5:10,-10:0.5:10,-10:0.5:10);

Result for Case 1:

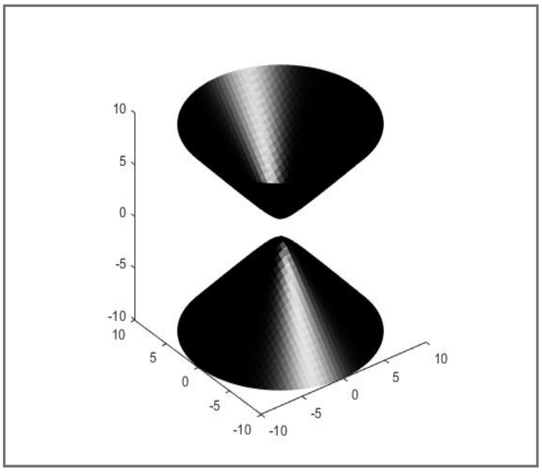

Fig. (1): Curvature two faces

Input:

Case 2: meshgrid (-10:0.5:10,-10:0.5:10,0:0.5:10);

Result for Case 2 :

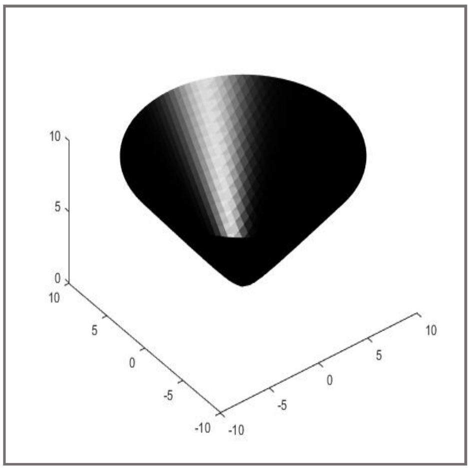

Fig.(2): Curvature positive face

Input:

Case 3: meshgrid (-10:0.5:10,-10:0.5:10,-10:0.5:0);

Result for Case 3

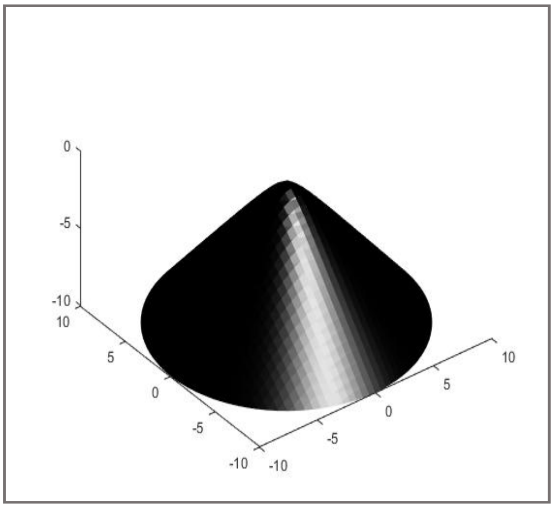

Fig. (3): Curvature negative face

MATLAP NORMAL CURVATURE

Code:

```
function [normals, curvature] = findPointNormals(points, numNeighbours, viewPoint, dirLargest)
validateattributes(points, {'numeric'},{'ncols',3});
if(nargin< 2)
numNeighbours = [];
end
if(isempty(numNeighbours))
numNeighbours = 9;
else
validateattributes(numNeighbours, {'numeric'},{'scalar','positive'});
if(numNeighbours> 100)
warning(['%i neighbouring points will be used in plane'...
' estimation, expect long run times, large ram usage and'...
' poor results near edges'],numNeighbours);
end
end
if(nargin< 3)
viewPoint = [];
end
if(isempty(viewPoint))
viewPoint = [0,0,0];
else
validateattributes(viewPoint, {'numeric'},{'size',[1,3]});
end
if(nargin< 4)
dirLargest = [];
end
if(isempty(dirLargest))
dirLargest = true;
else
validateattributes(dirLargest, {'logical'},{'scalar'});
end
points = double(points);
viewPoint = double(viewPoint);
kdtreeobj = KDTreeSearcher(points,'distance','euclidean');
n = knnsearch(kdtreeobj,points,'k',(numNeighbours+1));
n = n(:,2:end);
p = repmat(points(:,1:3),numNeighbours,1) - points(n(:),1:3);
```

```
p = reshape(p, size(points,1),numNeighbours,3);
C = zeros(size(points,1),6);

C(:,1) = sum(p(:,:,1).*p(:,:,1),2);
C(:,2) = sum(p(:,:,1).*p(:,:,2),2);
C(:,3) = sum(p(:,:,1).*p(:,:,3),2);
C(:,4) = sum(p(:,:,2).*p(:,:,2),2);
C(:,5) = sum(p(:,:,2).*p(:,:,3),2);
C(:,6) = sum(p(:,:,3).*p(:,:,3),2);

C = C ./numNeighbours;

normals = zeros(size(points));
curvature = zeros(size(points,1),1);
for i = 1:(size(points,1))
Cmat = [C(i,1) C(i,2) C(i,3);...
C(i,2) C(i,4) C(i,5);...
C(i,3) C(i,5) C(i,6)];
[v,d] = eig(Cmat);
d = diag(d);
[lambda,k] = min(d);
normals(i,:) = v(:,k)';
curvature(i) = lambda / sum(d);
end
points = points - repmat(viewPoint,size(points,1),1);
if(dirLargest)
[~,idx] = max(abs(normals),[],2);
idx = (1:size(normals,1))' + (idx-1)*size(normals,1);
dir = normals(idx).*points(idx) > 0;
else
dir = sum(normals.*points,2) > 0;
end
normals(dir,:) = -normals(dir,:);
end
```

Demo example:

```
x = repmat(1:49,49,1);
y = x';
z = peaks;
points = [x(:),y(:),z(:)];
[normals,curvature]
= findPointNormals(points,[],[0,0,10],true);
holdoff;
surf(x,y,z,reshape(curvature,49,49));
holdon;
quiver3(points(:,1),points(:,2),points(:,3),...
normals(:,1),normals(:,2),normals(:,3),'r');
axisequal;
```

Result:

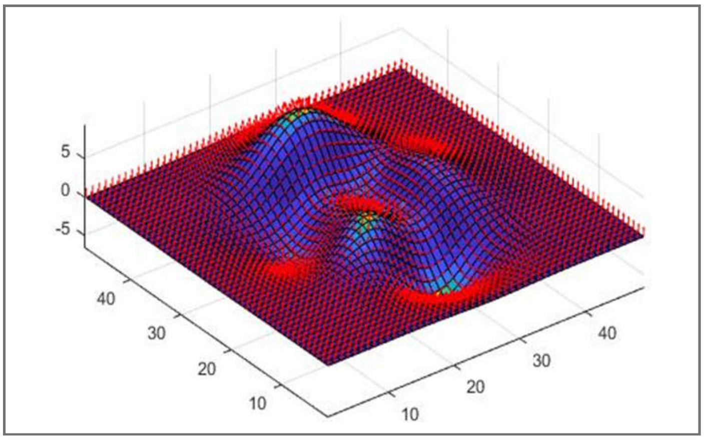

Fig. (4): Curvature normal

Results:

- In case 1 and case 2 and case 3 figures show the main results between two sheets curvature and one sheet curvature
- When we adjust the parameter of the equation

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (*)$$

in 3 Axis $[X, Y, Z]$ vectors as follows

Case 1: $x=[-10{:}\,0.5:10\,]$, $y=[-10{:}\,0.5:10\,]$, $x=[-10{:}\,0.5:10\,]$
in this case the equation $(*)$ show this parameters as two faces curvature and figure (6.1) show the result (two sheet)

Case 2 : $x=[-10{:}\,0.5:10\,]$, $y=[-10{:}\,0.5:10\,]$, $x=[0{:}\,0.5:10\,]$
In this case the equation $(*)$ show the parameter of case 2 as positive face of curvature (upper curvature)or one sheet curvature.

Case 3 : $x=[-10{:}\,0.\mathrm{curvature}5:10\,]$, $y=[-10{:}\,0.5:10\,]$, $x=[-10{:}\,0.5:0\,]$
in this case the equation $(*)$ processed the parameters in negative way of curvature (lower curvature) or one sheet curvature .
In normal curvature show the sample data in positive direction of plane IR^3 that means is approximately near to one sheet positive direction.

CONCLUSION

In this work, we have provided two different algorithms for curvature estimation comparison between Curvature and Normal Curvature And we found In case 1 , case 2 and case 3 figures show the main results between two sheets curvature and one sheet curvature when we adjust the parameter As for the normal curve shown the sample data in positive direction of plan R^3 that means is approximately near to case 2 direction.

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